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SUPERSYMMETRY, EXACTLY SOLVABLE PROBLEMS, AND NON-LINEAR ALGEBRAS

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Using basic methods of supersymmetric quantum mechanics we construct a new class of exactly solvable potentials having as supersymmetric partner potential that of the harmonic oscillator. The corresponding ladder operators are constructed and shown to close a non-linear algebra.

Since the early days of quantum mechanics there has been interest in exactly solvable quantum systems. Such systems are characterized by quantum mechanical Hamiltonians whose eigenvalues and eigenfunctions can be given in an explicit analytic and closed form. During the last two decades much progress has been made in the construction of such exactly solvable systems in one dimension.

The Darboux method, which is based on a note by Darboux[1] made in the last century, starts with an exactly solvable Hamiltonian H_+ and searches for a new Hamiltonian H_-, which is related to the previous one via a first-order differential operator $A^\dagger$ by the relation $A^\dagger H_+ = H_- A^\dagger$ and thus allows one to obtain the unknown H_--eigenfunctions from those of H_+. By construction H_+ and H_- are essentially isospectral.[2] This approach is closely related to the factorization method and supersymmetric (SUSY) quantum mechanics.[3] Recently the Darboux method has been extended to higher-order transformation operators $A^\dagger$ (including even an explicit time dependence) giving rise to non-linear superalgebras.[4]

Another different construction method for exactly solvable systems has been suggested by Abraham and Moses[5] and is based on the inverse method. As in the Darboux method one starts with a Hamiltonian whose spectral properties are exactly known and than constructs a new Hamiltonian with spectral properties following from those of the starting Hamiltonian. As a special case of this rather general method one can also recover results obtained via the Darboux method if applied to SUSY quantum systems.[6]

Recently, we have proposed yet a different method for constructing exactly solvable systems,[7] which is based on the supersymmetric formulation of quantum mechanics.[8] Here in essence one makes an ansatz for the so-called SUSY

potential such that one SUSY partner, say H_+, becomes one of the known exactly solvable systems. Then with the help of the SUSY transformation one can obtain the complete spectral properties of the other SUSY partner, which in general is some new exactly solvable quantum system. Here it turns out that the parameters characterizing the new system have to obey certain conditions and therefore these new systems have been called conditionally exactly solvable.

Below we will briefly present our main idea by discussing the case where H_+ is a harmonic oscillator Hamiltonian. For more details we refer the reader to the preprint in Ref.[7]. Following a suggestion of Mielnik[3] we also construct from the ladder operators of H_+ those corresponding to H_- and show that they obey a non-linear algebra which is of quadratic type.

Witten's model of SUSY quantum mechanics[8] is completely characterized by the SUSY potential W:

$$H_\pm = -\frac{1}{2}\frac{d^2}{dx^2} + V_\pm(x)\,, \qquad V_\pm(x) = \frac{1}{2}W^2(x) \pm \frac{1}{2}W'(x)\,. \tag{1}$$

With the SUSY transformation operator $A = (d/dx + W(x))/\sqrt{2}$ the two SUSY partner Hamiltonians factorize as follows: $H_+ = AA^\dagger$, $H_- = A^\dagger A$. Obviously, $A^\dagger H_+ = H_- A^\dagger$ and from the known eigenfunctions of H_+ one can find those of H_- for broken as well as unbroken SUSY.[7] It is also obvious that $W(x) = x$ corresponds to a SUSY pair of harmonic oscillators. Therefore, in order to find new potentials V_- we make the ansatz[7] $W(x) = x + u'(x)/u(x)$ and search for such functions u which give rise to a harmonic potential V_+. Indeed, if u is a solution of the differential equation

$$u''(x) + 2xu'(x) + 2(1-\varepsilon)u(x) = 0 \tag{2}$$

we have $V_+(x) = \frac{1}{2}x^2 + \varepsilon - \frac{1}{2}$ and the corresponding partner potential reads

$$V_-(x) = \frac{1}{2}x^2 - \varepsilon + \frac{1}{2} + \frac{u'(x)}{u(x)}\left(2x + \frac{u'(x)}{u(x)}\right). \tag{3}$$

The most general solution of (2) is given in terms of confluent hypergeometric functions

$$u(x) = \alpha\, {}_1F_1\!\left(\tfrac{1-\varepsilon}{2}, \tfrac{1}{2}, -x^2\right) + \beta x\, {}_1F_1\!\left(\tfrac{2-\varepsilon}{2}, \tfrac{3}{2}, -x^2\right). \tag{4}$$

Note that u must not have any zero in order to avoid singularities in V_-. Thus the potential parameters have to obey the conditions $\varepsilon > 0$ and $|\beta/\alpha| < 2\,\Gamma(\frac{1+\varepsilon}{2})/\Gamma(\frac{\varepsilon}{2})$. The above family of potentials (3) contains as special cases those which have already been found by the other methods. For example, for

$\alpha = \gamma$, $\beta = 0$ and $\varepsilon = 1$ we recover the result of Mielnik,[3] whereas for $\alpha = 1$, $\beta = 0$ and $\varepsilon = 2k + 1$ ($k \in \mathbb{N}$) we have the result of Bagrov and Samsonov,[4]

$$V_-(x) = \frac{x^2}{2} + 8k(2k-1)\frac{H_{2k-2}(ix)}{H_{2k}(ix)} - 16k^2\left(\frac{H_{2k-1}(ix)}{H_{2k}(ix)}\right)^2 + 2k - \frac{1}{2}, \quad (5)$$

where H_ν denotes a Hermite polynomial and $i^2 = -1$. Note that in essence the previous approaches correspond to the choice $\varepsilon = 1$ in the present one and, more generally, the cases $\varepsilon = 2k + 1$ are related to the k-order generalization of the Darboux method.[4]

Following Mielnik [3] we introduce the ladder operator $B = A^\dagger a A$ where $a = (d/dx + x)/\sqrt{2}$ is the usual one for the harmonic oscillator $H_+ = a^\dagger a + \varepsilon$. One can easily verify that this operator and its adjoint obey the relations

$$\begin{gathered}[H_-, B] = -B\,, \quad [H_-, B^\dagger] = B^\dagger\,, \quad [B, B^\dagger] = 3H_-^2 - (2\varepsilon - 1)H_-\,, \\ B^\dagger B = H_-(H_- - 1)(H_- - \varepsilon)\,, \quad BB^\dagger = (H_- + 1)H_-(H_- + 1 - \varepsilon)\,. \end{gathered} \quad (6)$$

Here we note that for the special case $\varepsilon = 3$ this quadratic algebra has recently been found by Spiridonov [9] who also noted that the ground state of H_- is isolated. This in fact is the case for all $\varepsilon > 0$.[7]

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